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Steady states of weakly dissipative systems

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Abstract. The states to which weakly dissipative systems in constant external conditions tend are examined. At vanishing dissipation, several steady states, which are superpositions of steady states of an isolated conservative system, can exist. With increasing dissipation, these states can form clusters. When the correlation time of the conservative motions is finite, a description with a probability density, which is a solution of a Fokker–Planck equation in a coarse grained phase space, seems possible.

1. Introduction

The deterministic dissipative problems which we discuss arise, mostly, from a description of physical systems with macroscopic variables: order parameters, gross or collective variables, modes, etc. In some cases the description may be partly microscopic, e.g. friction forces acting on heavy particles. 'Weakly' refers to a part of the dissipation to be defined later, the total dissipation may be important. The interaction to which the dissipation is due also produces fluctuating forces which we do not take into account, these systems produce their own 'macroscopic' fluctuations. An aspect of the opposite case, when the 'macroscopic' fluctuations are negligible, has been considered by Haken (1975a). We treat situations where the external conditions do not depend on time, with the aim of obtaining the physical properties of the state to which the system tends after a long time interval. This state is referred to as a steady state, although the macroscopic variables generally do not tend to constant values.

Steady states in which the variables do not vary in time are particularly well understood and they can often be interpreted in the framework of the thermodynamics of irreversible processes (Glansdorff and Prigogine 1971). There is also an extensive literature on steady states with variables periodic in time, based on methods related to the Hopf bifurcation. More complicated steady states form an object of the theories of turbulence in fluids, recent accounts are given by Leslie (1973), Orszag (1970), Monin and Yaglom (1971, 1974). We use a different approach and therefore do not review this work.

No consensus seems to exist about what dissipative systems are. We adopt the view that dissipation manifests itself in a qualitative property of the orbits: dissipative attraction. The orbits approach indefinitely asymptotically stable invariant sets in phase space, which are attractors if not decomposable. When a dissipative system is observed during a long time interval, the statistical properties of motions, which start in the domain of attraction (basin) of an attractor, are the same as those of motions which start on the attractor itself and remain on it. A probability distribution defined on an attractor thus also refers to its basin, to the entire phase space when the attractor is unique. For turbulent motions of fluids this has been formulated by Landau and Lifshitz (1969) and later by Hopf (1948); they thought the attractors were tori covered with quasi-periodic orbits. However, experiment indicates the existence of a continuous spectrum (Gollub and Swinney 1975). An explanation is due to Ruelle and Takens (1971) who showed that, in most cases, the orbits on a torus $\mathcal{T}^n$, $n \geqslant 4$ are not dense but have strange attractors which are not manifolds. The motions on these strange attractors have the expected qualitative properties.

Turbulent motions are probably frequent for dissipative systems with many degrees of freedom. It is likely that, in many cases, a dissipative equation of motion restricted to an invariant manifold, $n \geqslant 2$, is still dissipative: the orbits are not everywhere dense on the manifold but have attractors. The situation occurring on tori suggests that a typical turbulent attractor is probably strange.

Dissipative systems can be described with a statistical equation. Due to the volume contracting property of dissipative (semi)flows, the probability density may become singular when $t \to \infty$ and the Liouville equation cannot be used. An attempt to adapt this equation to dissipative systems was made by Hopf (1952). The characteristic function is used in place of the probability density, its evolution equation is the Fourier transform of the Liouville equation. For non-linear systems with many degrees of freedom Hopf's equation is very complicated and the stability of the steady state solutions must be checked.

Deterministic macroscopic equations give a somewhat incomplete description of dissipative systems, in fixed external conditions the steady state may not be unique. It becomes so when thermodynamic fluctuations are introduced. This question arises more rarely for weakly dissipative systems, as we shall see.

It would be desirable to formulate the theory of dissipative systems with the degree of generality of the thermodynamics of irreversible processes, but the steady states of strongly dissipative systems seem still very different from what they are in conservative systems. This is essentially due to the strangeness of the attractors. A probability distribution can still be defined, but it is not an invariant Lebesgue measure and this is a considerable complication.

It is therefore interesting that, when dissipation is weak, the steady states can be described by an approach which avoids these difficulties in most cases (Larisch 1977). It becomes possible due to the evolution of the attractors; when dissipation is vanishingly small, they are everywhere dense in much simpler sets. A coarse grained probability density can be defined when dissipation increases; it is the solution of a Fokker–Planck equation. The generally unique steady state appears as composed of several metastable states. The results can be formulated in the framework of the thermodynamics of irreversible processes and we shall present this in a forthcoming paper.

The idea of the approach is demonstrated for two simple examples. Then we discuss some qualitative properties of weakly dissipative systems before turning to their quantitative description.

2. Examples

2.1. The model of Saltzman and Lorenz

Since Lorenz (1963) performed numerical calculations and apparently found strange attractors for the equation (rescaled):

$$\dot{x}_1 = x_2 - \nu\alpha x_1 \qquad \dot{x}_2 = x_1(x_3 - 1) - \nu\beta x_2$$
$$\dot{x}_3 = -x_1 x_2 - \nu(x_3 - a) \qquad \alpha, \beta, \nu > 0 \tag{1}$$

which is a model for the transition to turbulence in convection; it has been studied by many authors who we cannot mention here. Haken (1975b) showed that it also describes certain lasers and that it refers to a conservative system with the equation of motion:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_1(x_3 - 1) \qquad \dot{x}_3 = -x_1 x_2 \tag{2}$$

under the action of a friction force $(-\nu\alpha x_1, -\nu\beta x_2, -\nu x_3)$ and of a driving force $(0, 0, \nu a)$. The conservative equation has two first integrals:

$$I_1 = x_1^2 + x_2^2 + x_3^2 \qquad I_2 = x_2 + (x_3 - 1)^2$$

The conservative equation (2) thus defines a decomposition of the phase space in closed invariant manifolds $\mathscr{J}(I)$ which are periodic orbits.

As noticed by Lorenz, the distance of a moving point from the origin decreases when it is outside the ellipsoid $\mathscr{P} : \dot{I}_1 = 0$. Any motion which starts at a finite distance from the origin will thus enter the smallest sphere $I_1 = f(\alpha, \beta, \nu)$ which encloses $\mathscr{P}$ after a finite time and remain there. One easily sees that $\mathscr{P}$ is a Poincaré section. Another qualitative property of equation (1) is that the basins of its attractors are infinite. A boundary of a basin, a separatrix $\mathscr{S}$, is an integral surface of equation (1) but this equation has no closed integral surfaces, as div $\dot{x} = -\nu(\alpha + \beta + 1) < 0$.

We are interested in the steady states of the model system when $\nu \ll 1$, so it is natural to look at deformations of those at $\nu \to +0$. New coordinates I_1, I_2, θ are introduced, where θ is an angle along $\mathscr{J}(I)$ and equation (1) becomes:

$$\dot{I}_1 = \nu D_{\perp 1}(I, \theta) \qquad \dot{I}_2 = \nu D_{\perp 2}(I, \theta) \qquad \dot{\theta} = C(I, \theta) + \nu D_{\parallel}(I, \theta). \tag{3}$$

When $\nu \ll 1$, one may expect a relation between the attractors $\mathscr{A}^\nu$ of equation (3) and some of the $\mathscr{J}(I)$, $\lim_{\nu \to +0} \mathscr{A}^\nu = \mathscr{J}(I_\alpha)$ (e.g., Urabe 1967, Moser 1970 treats conservative systems but his approach is closer to ours). To find these I_α, equation (3) can be treated with the method of averaging:

$$\dot{I} \cong \frac{\nu}{T(I)} \int_0^{T(I)} D_\perp(I, \theta)\, \mathrm{d}t = \frac{\nu}{T(I)} \int_0^{2\pi} D_\perp(I, \theta) \frac{\mathrm{d}\theta}{C(I, \theta)}. \tag{4}$$

The average is for the conservative motion on $\mathscr{J}(I)$ so I is kept constant. $D_\perp$ is the component of D normal to $\mathscr{J}(I)$. The quantity $\phi_{\mathscr{J}(I)} = (TC)^{-1}$ is the conservative probability density and equation (4) can be written:

$$\dot{I} = \nu \langle D_\perp(I, \theta) \rangle_{\mathscr{J}(I)} \tag{5}$$

where $\langle \ldots \rangle_{\mathscr{J}(I)}$ denotes conservative averaging on $\mathscr{J}(I)$.

The I_α are the isolated attracting critical points of equation (5). When $\nu \ll 1$, $\nu < \nu^*$, $\mathscr{A}^\nu$ is a periodic orbit situated in a ν-neighbourhood of $\mathscr{J}(I_\alpha)$. Thus, at $\nu < \nu^*$ the model system can have several steady states S_α^ν which are close to 'microcanonic' states S_α of the conservative system. All attractors of equation (5) are probably points but some may not be isolated; they would correspond to turbulent attractors.

Qualitative changes occur when $\nu > \nu^*$. Let us assume there are two attractors, at $\nu > \nu_1^*$, ν_2^* they can coalesce. Then, an orbit on $\mathscr{A}^\nu$ remains, on the average, a proportion of the time p_1, p_2 near the attractors $\mathscr{A}_1^0$, $\mathscr{A}_2^0$ and the steady state can be approximately characterised as $S^\nu \sim p_1 S_1 + p_2 S_2$. Such a situation has been described by Haken and Wunderlin (1977).

2.2. *The model of Lefever and Prigogine (1968)*

This model has been introduced as a simple example of chemical reaction-diffusion leading to dissipative structures. It has been studied by several authors, more recently by Auchmuty and Nicolis (1976). After a rescaling, the evolution equation is:

$$\begin{aligned} \dot{u}_1 &= u_1^2 u_2 - u_1 - \nu(u_1 - a - d_1\,\Delta u_1) \\ \dot{u}_2 &= -(u_1^2 u_2 - u_1) + \nu d_2\,\Delta u_2 \end{aligned} \tag{6}$$

$u = u(x, t)$; u, ν, a, d_1, $d_2 > 0$. An initial and boundary value problem is formulated for this equation. Again, we shall be interested in the properties of the stable steady states when $\nu \ll 1$.

When $\nu = 0$, equation (6) has the first integrals $2I = u_1 + u_2$. In the variable I, $2\theta = u_1 - u_2$:

$$\begin{aligned} I &= \tfrac{1}{2}\nu[-I - \theta + a + (d_1 + d_2)\,\Delta I + (d_1 - d_2)\,\Delta\theta] \\ \theta &= -(\theta + I)(\theta^2 - I^2 + 1) + \tfrac{1}{2}\nu[-I - \theta + a - (-d_1 + d_2)\,\Delta I - (d_1 + d_2)\,\Delta\theta]. \end{aligned}$$

Thus, when $\nu = 0$, the evolution equation defines a decomposition of the phase space in invariant manifolds which extend to infinity, the planes $u_1 + u_2 = \text{constant}$. On each of these planes the motion is dissipative; it has the attractor $\theta = -I$ and, for $I > 1$, also the attractor $\theta = (I^2 - 1)^{1/2}$. We consider situations when not only $\nu \ll 1$ but also $\nu d\,\Delta u \ll 1$, excepting narrow transition layers. Then, in the bulk of the recipient, the evolution can be described as follows. A rapid motion brings the system to the neighbourhood of one of the above attractors. Then, the system evolves slowly according to one of the equations:

$$\dot{I} = \tfrac{1}{2}\nu(a + 2d_2\,\Delta I)$$

and, for $I > 1$, also;

$$\dot{I} = \tfrac{1}{2}\nu[a - I - (I^2 - 1)^{1/2} + (d_1 + d_2)\,\Delta I + (d_1 - d_2)\,\Delta(I^2 - 1)^{1/2}].$$

We cannot discuss here the variety of steady states which occur for different boundary conditions and values of a, d.

3. Some qualitative properties of dissipative systems

We consider discrete systems with a finite-dimensional phase space, although many results can be easily extended to continuous systems. As a definition for 'dissipative' we take the basic property of equation (1): to an initial condition corresponds a unique motion which, after a finite time, enters some closed domain of the phase space and remains there. Obviously, all attractors of such systems are finite.

Let $\dot{x} = X(x)$ be the equation of motion of the system. A situation frequently met is that the dissipative vector field X is 'hyperbolic': it is contracting in some directions and dilating in others. Hyperbolic vector fields have a practically important property: some, at least, of their attractors are on invariant manifolds (Neimark 1972). No general method to find these invariant manifolds seems to exist. When X is contracting in all directions, there is a unique attractor, a critical point.

In many cases the (semi)flow of X is volume contracting (div $X < 0$). Then, as for the Saltzman–Lorenz model, the phase space, and also the basins of the attractors, are infinite; indeed, no closed separatrix can exist.

When X has a symmetry group g, the phases x_g with this symmetry are on invariant manifolds $\mathscr{I}_g$ which decompose in parts corresponding to the irreducible subgroups g_i of g. Each manifold $\mathscr{I}_{g_i}$ has attractors of the flow of X restricted to $\mathscr{I}_{g_i}$, and some of these 'restricted' attractors may actually have a basin of non-zero volume. When this occurs, almost symmetric vector fields may be expected to have almost symmetric attractors, due to the structural stability of dissipative systems. Attractors with a symmetry group are often observed experimentally.

Little is known about the nature and number of the attractors of dissipative systems, in general. The weakly dissipative systems however, to which we turn now, can be described in detail. These systems have equations of motion which resemble equation (1):

$$\dot{x} = X(x) = C(x) + \nu D(x) \qquad x = (x_1, \ldots, x_N) \tag{7}$$

where C is a conservative and D a dissipative vector field. Conservative here means the following. The vector field C has at least one first integral $\{\mathscr{I}_1\}$ whose integral surfaces $\mathscr{I}_1$ are closed. Let $\{\mathscr{I}_1\}, \ldots, \{\mathscr{I}_n\}$ be the $n \leqslant N$ first integrals of C; the intersection of surfaces from all first integrals defines a family of invariant manifolds of C, $\{\mathscr{I}(I)\}$, which are numbered by a vector $I = (I_1, \ldots, I_n)$. The flow of C restricted to most $\mathscr{I}(I)$ is assumed ergodic. Obviously, if the first integrals can be written as $F_k(x) = I_k$, $k = 1, \ldots, n$, $\mathscr{I}(I)$ is the manifold where $F(x) = I$. As we shall see, the properties of these systems also resemble those of the model of Saltzman and Lorenz.

The steady states of systems with such equations of motion, if the $\mathscr{I}(I)$'s are tori and the orbits of C are quasi-periodic, can sometimes be determined exactly when $\nu \ll 1$ (see, e.g., Mitropolski and Lykova 1973 for results and references). As in § 2, coordinates θ are introduced on the manifolds $\mathscr{I}(I)$ and equation (7) is written in the form:

$$\dot{I} = \nu D_\perp(I, \theta) \qquad \dot{\theta} = C(I, \theta) + \nu D_\parallel(I, \theta) \tag{8}$$

The components of D normal and parallel to the manifold $\mathscr{I}(I)$ are denoted $D_\perp$, $D_\parallel$; one has $D_\perp = D$ grad F. To equation (8) corresponds the averaged equation:

$$\dot{I} = \nu \langle D_\perp(I, \theta)\rangle_{\mathscr{T}(I)} = \nu D_\perp^0(I). \tag{9}$$

The average $\langle \ldots \rangle_{\mathscr{T}(I)}$ is calculated with the probability distribution on the torus $\mathscr{T}(I)$ of the conservative flow—the 'microcanonic' distribution. If the averaged equation (9) has an attracting isolated critical point I_α then, for $0 < \nu < \nu^*(I_\alpha)$, equation (8) has an invariant asymptotically stable torus $\mathscr{T}^\nu(I_\alpha)$ in a ν-neighbourhood of $\mathscr{T}(I_\alpha)$ and $\lim_{\nu \to +0} \mathscr{T}^\nu(I_\alpha) = \mathscr{T}(I_\alpha)$. The torus $\mathscr{T}^\nu$ itself is unlikely to be an attractor when $N - n > 1$. If $N - n = 2$, in most cases the attractors are critical and periodic points; when $N - n \geqslant 4$, they are likely to be strange, as mentioned.

No proof seems to exist when the attractor of the averaged system is not a point or when the conservative orbits on the $\mathscr{I}(I)$ are not quasi-periodic. Such orbits are related to completely integrable Hamiltonian systems and are thus rare. We shall use a conjecture introduced by Larisch (1977) which applies to dissipative systems with equations of motion having the form (8) and the ergodic property of the conservative flow. The averaged equation:

$$\dot{I} = \nu \langle D_\perp(I, \theta) \rangle_{\mathscr{I}(I)} = \nu D_\perp^0(I) \tag{10}$$

is obtained using for the calculation of $\langle \ldots \rangle_{\mathscr{I}(I)}$ the probability distribution on $\mathscr{I}(I)$, again the 'microcanonic' probability distribution. When div $C = 0$, as is the case for Hamiltonian systems, the probability density $\phi_{\mathscr{I}(I)}$ is well known. The volume in phase space is an invariant measure, so if $\mathrm{d}\tau = \tilde{\phi}(I, \theta)\, \mathrm{d}I\, \mathrm{d}\theta$ is the volume element,

$$\phi_{\mathscr{I}(I)}(\theta) = \tilde{\phi}(I, \theta) \left(\int_{\mathscr{I}(I)} \tilde{\phi}(I, \theta)\, \mathrm{d}\theta \right)^{-1} \tag{11}$$

and

$$D_\perp^0(I) = \int_{\mathscr{I}(I)} D_\perp(I, \theta) \phi_{\mathscr{I}(I)}(\theta)\, \mathrm{d}\theta. \tag{12}$$

Let $\mathscr{D}$ be an attractor of the averaged flow defined by equation (10). We denote by $\mathscr{I}(\mathscr{D})$ the set in phase space made of $\mathscr{I}(I)$'s with I in $\mathscr{D}$. It is very likely that, when ν is small, there are attractors $\mathscr{A}^\nu$ in a ν-neighbourhood of the $\mathscr{I}(\mathscr{D})$'s. The conjecture is that when $\nu \to +0$ the attractors $\mathscr{A}^0$ are everywhere dense in the $\mathscr{I}(\mathscr{D})$'s; the ergodicity of the conservative flow on the $\mathscr{I}(I)$ makes this plausible. As we shall see, this conjecture can be brought closer to a proof.

One could imagine that equation (8) has asymptotically stable invariant sets $\mathscr{I}^\nu(\mathscr{D})$ near the $\mathscr{I}(\mathscr{D})$ and $\lim_{\nu \to +0} \mathscr{I}^\nu(\mathscr{D}) = \mathscr{I}(\mathscr{D})$ but this does not necessarily occur. When an attractor $\mathscr{A}^\nu$ is strange and there are several attractors $\mathscr{D}$, $\mathscr{A}^\nu$ may decompose into several 'weakly' connected pieces which are near different $\mathscr{I}(\mathscr{D})$'s. When $\nu \to +0$ these pieces are everywhere dense in the corresponding $\mathscr{I}(\mathscr{D})$.

The important case when there is one conservative first integral is particularly simple. In this case $\mathrm{d}\tau = \mathrm{d}\sigma\, \mathrm{d}I_1/\|\nabla F_1\|$ where $\mathrm{d}\sigma$ is the area element on $\mathscr{I}(I_1)$, so the 'microcanonic' probability distribution is:

$$\mathrm{d}\phi_{\mathscr{I}(I_1)} = \left(\int_{\mathscr{I}(I_1)} \frac{\mathrm{d}\sigma}{\|\nabla F_1\|} \right)^{-1} \frac{\mathrm{d}\sigma}{\|\nabla F_1\|} \tag{13}$$

and the equation of the averaged flow is:

$$\dot{I}_1 = \nu \left(\int_{\mathscr{I}(I_1)} \|\nabla F_1\|^{-1}\, \mathrm{d}\sigma \right)^{-1} \int_{V(\mathscr{I}(I_1))} \operatorname{div} D\, \mathrm{d}\tau. \tag{14}$$

The last integral is over the volume enclosed by the surface $\mathscr{I}(I_1)$. In the frequently met case when $\langle \operatorname{div} D \rangle_{\mathscr{I}(I_1)} < 0$ the attractors $\mathscr{D}$ are the points $I_{1\alpha}$ for which $V = 0$; the $\mathscr{I}(\mathscr{D})$'s are thus the degenerate surfaces of the first integral.

4. Steady states of weakly dissipative systems

A steady state S^ν of a dissipative system is naturally identified with an attractor $\mathscr{A}^\nu$ of its equation of motion. The converse may not be true in weakly dissipative systems; several attractors $\mathscr{A}^\nu$ may correspond to the same steady state S^ν, that is have close statistical properties. A physical property can be written as:

$$u = \langle U(x) \rangle_{\mathscr{A}} = \int_{\mathscr{A}} U(x)\, \mathrm{d}\mu_{\mathscr{A}} \tag{15}$$

where $d\mu_{\mathscr{A}}$ is the probability distribution on $\mathscr{A}$. The computation of this average is still difficult; due to the frequent strangeness of $\mathscr{A}$, $d\mu_{\mathscr{A}}$ is not a Lebesgue measure.

It is important to remark that what one actually needs for the description of the steady state is not necessarily the attractor $\mathscr{A}$ and the probability distribution $d\mu_{\mathscr{A}}$, but a procedure for the calculation of physical properties. Suppose we can immerse the attractor $\mathscr{A}$ in a manifold $\tilde{\mathscr{A}}$ and define on $\tilde{\mathscr{A}}$ a measure $d\mu_{\tilde{\mathscr{A}}}$ so that:

$$\int_{\mathscr{A}} U(x)\, d\mu_{\mathscr{A}} \cong \int_{\tilde{\mathscr{A}}} U(x)\, d\mu_{\tilde{\mathscr{A}}} \tag{16}$$

for any smooth function $U(x)$. Then the statistics of the steady state could be based on $\mu_{\tilde{\mathscr{A}}}$ and this would be a considerable simplification.

A known, but complicated, way to perform this is to assume that the system interacts weakly ($\ll \nu$) with a reservoir. The action of the reservoir is a small random force added to X. A probability density ϕ, which is a solution of a Fokker–Planck equation, can be introduced. The attractors are supposed stable to such perturbations (Sinai 1972 proved this in some cases) so ϕ will have very marked maxima in regions which narrowly enclose the attractors. These regions can be used as manifolds $\tilde{\mathscr{A}}$ with $d\mu_{\tilde{\mathscr{A}}} = \phi\, dx$. Haken (1975a) showed that when the interaction with the reservoir is of the order of ν the Fokker–Planck equation can be reduced to one for a system whose equation of motion is (10).

The conjecture introduced in § 3 permits us to avoid the artifice of a reservoir. We shall look first at the limit $\nu \to +0$. Since the attractors $\mathscr{A}^0$ are assumed to be dense everywhere in the sets $\mathscr{J}(\mathscr{D})$ these sets can be used as a support for a probability distribution (Larisch 1977). Conservative systems with more than two first integrals are probably rare, so the $\mathscr{J}(\mathscr{D})$ are in most cases manifolds, critical or periodic points.

The probability distribution for the steady states S^0 can be easily obtained. Let $d\psi_{\mathscr{J}(\mathscr{D})}$ be the probability distribution for the steady states with attractors which are in $\mathscr{J}(\mathscr{D})$. The probability distribution of I is defined by the averaged equation (10) and does not depend on θ, so $d\psi_{\mathscr{J}(\mathscr{D})}$ can be written as a product:

$$d\psi_{\mathscr{J}(\mathscr{D})} = d\psi_{\mathscr{J}(I)}\, d\psi_{\mathscr{D}} \tag{17}$$

where the first factor is the probability distribution of θ when I is fixed and the second factor is the probability distribution of I. $d\psi_{\mathscr{J}(I)}$ refers to the flow of $C + \nu D_{\|}$ on the $\mathscr{J}(I)$ in limit $\nu \to +0$; as the flow of C is ergodic, this limit does not depend on $D_{\|}$ and is simply the microcanonic distribution. When $\mathscr{D}$ is a manifold, that is in most cases,

$$d\psi_{\mathscr{D}} = \phi_{\mathscr{D}}(I)\, dI. \tag{18}$$

The easiest to describe are $\mathscr{D}$'s which are isolated critical points I_α, and $d\psi_{\mathscr{J}(\mathscr{D})} = d\psi_{\mathscr{J}(I_\alpha)}$. The system is in a 'microcanonic' state with the conservative first integrals having the values I_α. Such steady states, if $n = 1$, have properties of thermodynamic equilibrium. Many examples are known, an early one has been observed for tunnel diodes (Landauer 1962).

When $\mathscr{D}$ is a periodic orbit, $I = I(\sigma)$, $I(\sigma + 1) = I(\sigma)$, as in § 2:

$$d\psi_{\mathscr{D}} = \left(\int_0^1 \frac{d\sigma}{\|D_\perp^0(I(\sigma))\|} \right)^{-1} \frac{d\sigma}{\|D_\perp^0(I(\sigma))\|}. \tag{19}$$

The qualitative physical properties of a system with a periodic $\mathscr{D}$ depend on the nature of the conservative orbits. When they are quasi-periodic, a slow periodic variation of the frequencies appears. This effect can be observed in systems of weakly coupled harmonic oscillators with small friction and driving forces. In other cases, a periodic process is superposed on a random background. This is, probably, observed in turbulent boundary layers (for a review see Willmarth 1975).

A system whose attractor $\mathscr{D}$ is not a point can be described as undergoing an almost adiabatic process whose time variation is that of the motion of I on $\mathscr{D}$. The effect of this slow modulation on the line spectrum of a system with quasi-periodic orbits is well known.

We turn now to situations where ν cannot be considered vanishingly small. The properties of systems where C has quasi-periodic orbits should not change significantly as long as ν remains sufficiently small. It is probable that, in this case, there are attractors $\mathscr{A}^\nu$ in a ν-neighbourhood of a $\mathscr{J}(\mathscr{D})$ even when $\mathscr{D}$ is not a point, if $\nu<\nu^*$. Then, for $\nu<\min\nu^*$, one may expect the probability distribution to be close to that for $\nu=+0$. Close here means that the difference in simultaneous averages is small with ν, but qualitative differences can appear.

A completely different situation occurs when the conservative flow on the $\mathscr{J}(I)$ has finite correlation time. Let $t_c(I)$ be the correlation time on $\mathscr{J}(I)$ defined by:

$$\langle U(\theta)\rangle_{\mathscr{J}(I)}=0,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t U(\theta(t'))U(\theta(t'+\tau))\,\mathrm{d}t'\approx 0 \qquad \tau>t_c(I). \tag{20}$$

Then, when

$$\nu \ll (\max t_c(I))^{-1} \qquad I \text{ on } \mathscr{D} \tag{21}$$

as we shall see, $\nu^*=+0$. To put this in evidence, we write the equation of motion in the form:

$$\dot{I}=\nu D_\perp^0+\nu(D_\perp-D_\perp^0)=\nu D_\perp^0(I)+\nu D'_\perp(I,\theta) \qquad \dot{\theta}=C+\nu D_\parallel. \tag{22}$$

This equation can be interpreted as describing the interaction of the 'collective' variables I with the 'microscopic' variables θ due to dissipation. The effect of this interaction on the collective motion is through the fluctuating force $D'_\perp$. When ν is as in (21), the evolution of the collective variables is like a Brownian motion in the force field $\nu D_\perp^0$. Actually, it resembles a diffusion and the orbit of I will leave the ν-neighbourhood of the $\mathscr{J}(\mathscr{D})$ already when $\nu>0$.

A precise description of such a system is too complicated, the attractors can be imagined as being spread over the phase space. To obtain a smooth picture, we use a coarse graining of the phase space in bundles of $\mathscr{J}(I)$'s of width δI with:

$$\nu t_c\|D_\perp\| \ll \delta I \ll d(\mathscr{J}(I)) \tag{23}$$

where $d(\mathscr{J}(I))$ is the diameter of $\mathscr{J}(I)$. Rather than in the fine structure of the attractors, we are interested in the variation of $\int \mathrm{d}\mu_{\mathscr{A}}$, where the integral is over a bundle; we represent it with a probability density $\phi(I)$: $\int \mathrm{d}\mu_{\mathscr{A}}=\int \mathrm{d}\psi_{\mathscr{J}(I)}\,\phi(I)\,\mathrm{d}I$. A master equation can be written for $\phi(I)$ (see e.g. Stratonovich 1963) which, when truncated, reduces to the Fokker–Planck equation:

$$-\nu\frac{\partial}{\partial I_i}[(D_{\perp i}^0+\nu d_i)\phi]+\frac{\nu^2}{2}\frac{\partial^2}{\partial I_i\,\partial I_k}(\kappa_{ik}\phi)=0. \tag{24}$$

The additional drift coefficient $\nu^2 d$ is given by:

$$d_i = \int_{-\infty}^{0} K\left[\frac{\partial D_{\perp i}}{\partial I_k}, T^\tau D_{\perp k}\right] d\tau \tag{25}$$

and the diffusion coefficient $\nu^2 \kappa$ is:

$$\kappa_{ik} = \int_{-\infty}^{0} K[D_{\perp i}, T^\tau D_{\perp k}]\, d\tau. \tag{26}$$

The notations are:

$$T^\tau U(I(t), \theta(t)) = U(I(t)\theta(t+\tau))$$
$$K[U(I, \theta), V(I, \theta)] = \langle UV\rangle_{\mathscr{I}(I)} - \langle U\rangle_{\mathscr{I}(I)}\langle V\rangle_{\mathscr{I}(I)}.$$

We can now formulate precisely the notion 'weakly dissipative'. When the conservative orbits are quasi-periodic, it means $\nu < \nu^*$; when they have finite correlation time, it means the possibility of coarse graining expressed by the inequalities (23).

The Fokker–Planck equation (24) is supposed to have a unique acceptable solution, with finite norm and vanishing at the boundary of the system (the Fokker–Planck operator is contracting, Lebowitz and Bergman 1957). The effect of weak dissipative forces on a conservative system whose motions have finite correlation time is thus to create a unique steady state S^ν. This state has a 'fine structure'; it is composed of several steady states S^ν_α with ν-close statistical properties, corresponding to the different $\mathscr{A}^\nu_\alpha$. The probability density ϕ has very marked maxima in ν-neighbourhoods of the attractors $\mathscr{D}$; practically, it is only near the sets $\mathscr{I}(\mathscr{D})$ that the system can be found. An intuitive picture is that the attractors $\mathscr{A}^\nu_\alpha$, which are in the $\mathscr{I}(\mathscr{D})$ when $\nu = +0$, grow protuberances when ν increases, which spread over the phase space and connect the different $\mathscr{I}(\mathscr{D})$. The coalescence of the attractors, which can occur at $\nu > \nu^*$ in the quasi-periodic case, starts here presumably at $\nu > 0$.

The physical properties can be calculated with the solution of equation (24):

$$u = \int \langle U(I, \theta)\rangle_{\mathscr{I}(I)}\phi(I)\, dI. \tag{27}$$

More precisely, S^ν is unique should mean:

$$\int \langle U\rangle_{\mathscr{I}(I)}\phi(I)\, dI - \int_{\mathscr{A}} V\, d\mu_{\mathscr{A}} = O(\nu^2) \tag{28}$$

for almost all attractors $\mathscr{A}$—those with a non-zero basin when $\nu \to +0$. Thus, the probability distribution $\phi\, dI$ plays, for weakly dissipative systems, the role of the canonical distribution for conservative systems.

The composition of the states S^ν can be described as follows. Each attractor $\mathscr{D}_\alpha$ can be characterised by a probability p_α:

$$p_\alpha = \int \phi\, dI \qquad \sum p_\alpha \cong 1. \tag{29}$$

The integral is over a domain which includes only $\mathscr{D}_\alpha$ and so that ϕ is small on its boundary. It can also be characterised by a relaxation time τ_α, and $p_\alpha/\tau_\alpha = \text{constant}$.

The state S^ν is a superposition of the metastable states S^0_α:

$$S^\nu \cong \sum p_\alpha S^0_\alpha = \sum \tau_\alpha S^0_\alpha \tag{30}$$

As is well known for such cases, metastable states with a relaxation time comparable to the duration of the experiment appear as stable. When $\nu \to +0$, the relaxation times become infinite so the transition between the metastable states is impossible.

We assumed that the Fokker–Planck equation has a unique acceptable solution, but this imposes conditions on the diffusion coefficient which may not always be fulfilled. For example, among the $\mathscr{I}(I)$ a small number may not have finite correlation time. If this occurs on a $\mathscr{I}(I)$, it occurs generally also on the $\mathscr{I}(I)$'s in its neighbourhood. In these small volumes the diffusion coefficient cannot be defined and they may exert some influence on the solution of the Fokker–Planck equation. If these volumes are small in one direction only, they may constitute barriers to diffusion and produce a decomposition of the solution of the Fokker–Planck equation in several independent parts.

5. Discussion

The approach, presented for systems with equations of motion like the Saltzman and Lorenz model, can easily be extended to those resembling the model of Prigogine and Lefever. Fluids in turbulent motion could be an example. We still consider systems with equations of motion of the form (7), but the assumed properties of C are different. The requirements that C had a closed first integral and that its flow restricted to the $\mathscr{I}(I)$'s be ergodic is dropped. Instead, this flow could be dissipative with finite attractors $\mathscr{C}^\alpha(I)$. The results of §§ 3, 4 can easily be extended to such cases.

To simplify, let there be one attractor $\mathscr{C}(I)$; then, we replace the conservative average on $\mathscr{I}(I)$ with an average on $\mathscr{C}(I)$:

$$\dot{I} = \nu\langle D_\perp\rangle_{\mathscr{C}(I)} + \nu D'_\perp(I, \theta). \tag{31}$$

Due to the presumed structural stability of the dissipative flow restricted to the $\mathscr{I}(I)$'s, averages calculated on the attractors of $C + \nu D_\parallel$ will tend to those on the $\mathscr{C}(I)$'s when $\nu \to 0$. The Fokker–Planck equation, corresponding to equation (31) can also be written, if the flow of C restricted to the $\mathscr{C}(I)$'s has finite correlation time.

The diffusion approximation for the probability density, expressed by equation (24), is obtained from a formal expansion of the stochastic process in powers of ν, in which only the first two terms are kept, while higher derivatives, multiplied with higher powers of ν are neglected. The validity of the approximation can be checked in some cases. It seems plausible, but a result characterising the systems for which the Fokker–Planck equation gives a correct qualitative picture at small ν is lacking. Expansions of this type, in a slowness parameter, have been used, e.g., by Mori *et al* (1974). When the diffusion approximation applies, it is a confirmation of the conjecture introduced in § 3; indeed, when $\nu \to +0$, the support of the probability density contracts to the $\mathscr{I}(\mathscr{D})$'s.

We have shown that when the statistical properties of the system with the equation of motion $x = C$ are known, they can easily be obtained also when there is a weak dissipative action. Indeed, it has been possible to formulate an independent problem for the statistical distribution of the values of the first integrals. This is similar to the theories in which the dynamics of a part of the components of the phases of a system is studied. Green (1952) and Van Kampen (1957) use the conservative first integrals; Prigogine (1962) treats a similar problem when the perturbation is also conservative; as an application of the Nakashima–Zwanzig method we mention Nordholm and Zwanzig (1975), for Mori's method, Mori and Fujisaka (1973). For dissipative systems, methods based on an evolution equation for the probability density in phase space meet with the difficulties already mentioned in using such an equation. In the weakly dissipative case, and in the steady state, a coarse grained probability density in a reduced phase space can be defined and an equation for it written.

The ergodic theory of dissipative flows also lacks, apparently, general results and it is not always possible to say if the basic relation:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t F(x(t')t\,\mathrm{d}t' = \int_{\mathscr{A}} F(x)\,\mathrm{d}\mu_{\mathscr{A}}$$

with an orbit starting in the basin of an attractor, but not on it, is true. When $\mathscr{A}$ has an even number of invariant parts, the invariant measure may not be unique and additional conditions are then necessary to choose the ergodic measure; these conditions are known in some cases (e.g. for the so called axiom A flows, Bowen and Ruelle 1975).

The description of the steady states of weakly dissipative systems, which appears from § 4, seems particularly simple. When $\nu \to +0$, there can be several steady states S^0 in which 'microcanonic' states are superposed on the probability distribution of the attractors $\mathscr{D}$. For systems with a quasi-periodic conservative flow, this picture is only slightly modified when $\nu < \nu^*$; at $\nu > \nu^*$ the steady states can form clusters through a coalescence of attractors. Conservative flows with finite correlation time induce a fusion of the S^0 steady states already when $\nu > 0$. In this case a generalised canonical probability density can be defined which is a solution of a Fokker–Planck equation. We hope that further research will show that this simple and intuitive picture covers a broad range of dissipative systems.

References

Auchmuty J F J and Nicolis G 1975 *Bull. Math. Biol.* **37** 323
Bowen R and Ruelle D 1975 *Inventiones Math.* **29** 181
Glansdorff P and Prigogine I 1971 *Thermodynamic Theory of Structure, Stability and Fluctuations* (New York: Wiley)
Gollub J P and Swinney H L 1975 *Phys. Rev. Lett.* **35** 927
Green M S 1952 *J. Chem. Phys.* **245** 140
Haken H 1975a *Rev. Mod. Phys.* **47** 62
—— 1975b *Phys. Lett.* **53A** 77
Haken H and Wunderlin A 1977 *Phys. Lett.* **62A** 133
Hopf E 1948 *Commun. Pure Appl. Math.* **1** 303
—— 1952 *J. Ration. Mech. Analysis* **1** 87
Landau L D and Lifshitz E M 1969 *Fluid Mechanics* (Oxford: Pergamon)
Landauer R 1962 *J. Appl. Phys.* **33** 2309
Larisch E 1977 *Phys. Lett.* **62A** 207
Lebowitz J L and Bergmann P G 1957 *Ann. Phys., NY* **1** 1
Lefever R and Prigogine I 1968 *J. Chem. Phys.* **48** 1696
Leslie D C 1973 *Rep. Prog. Phys.* **36** 1365
Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
Mitropolski Y A and Lykova P 1973 *Integral Manifolds in Nonlinear Mechanics* (Moscow: Nauka)

Monin A S and Yaglom A M 1971 *Statistical Fluid Mechanics* vol. 1 (Cambridge, Mass.: MIT Press)
—— 1974 *Statistical Fluid Mechanics* vol. 2 (Cambridge, Mass.: MIT Press)
Mori H and Fujisaka H 1973 *Prog. Theor. Phys.* **49** 764
Mori H, Jugisaka H and Shigematsu H 1974 *Prog. Theor. Phys.* **51** 109
Moser J 1970 *Commun. Pure. Appl. Math.* **23** 609
Neimark Y A 1972 *The Method of Mappings in the Theory of Nonlinear Oscillations* (Moscow: Nauka)
Nordholm K S J and Zwanzig R 1975 *J. Statist. Phys.* **13** 347
Orszag S A 1970 *Statistical Theory of Turbulence* (Cambridge: Cambridge University Press)
Prigogine I 1962 *Non-Equilibrium Statistical Mechanics* (New York: Interscience)
Ruelle D and Takens F 1971 *Commun. Math. Phys.* **20** 127
Sinae Y G 1972 *Russ. Math. Surv.* **166** 21
Stratonovich R L 1963 *Topics in the Theory of Random Noise* vol. 1 (New York: Gordon and Breach)
Urabe M 1967 *Nonlinear Autonomous Oscillations* (New York: Academic)
Van Kampen N G 1957 *Physica* **23** 707
Willmarth W W 1975 *Advances in Applied Mechanics* vol. 15 (New York: Academic)